BLACK-COX MODEL

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Black and Cox extend Merton's model by assuming that default actually can happen before the maturity date. Many extensions of this model called First Passage Time model by Leland (1994), Briys and de Varenne (1997), Brigo and Tarenghi (2004). Figure 0.1 illustrates the difference between Merton's model and Black-Cox model in timing of the default.

Wiener process is continuous in time and composition of continuous functions is still continuous. We assume $A_0 > L$ and the asset of the firm follows GBM, i.e., $dA_t = \mu A_t dt + \sigma A_t dZ_t$. In case you are new to Itô's lemma, I will give the reason why the solution of A_t is a Geometric Brownian motion here. Applying Itô's lemma to $\ln A_t$, we can get

$$\begin{aligned} d\ln A_t &= \frac{1}{A_t} dA_t - \frac{1}{2A_t^2} (dA_t)^2 \\ &= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dZ_t \end{aligned}$$

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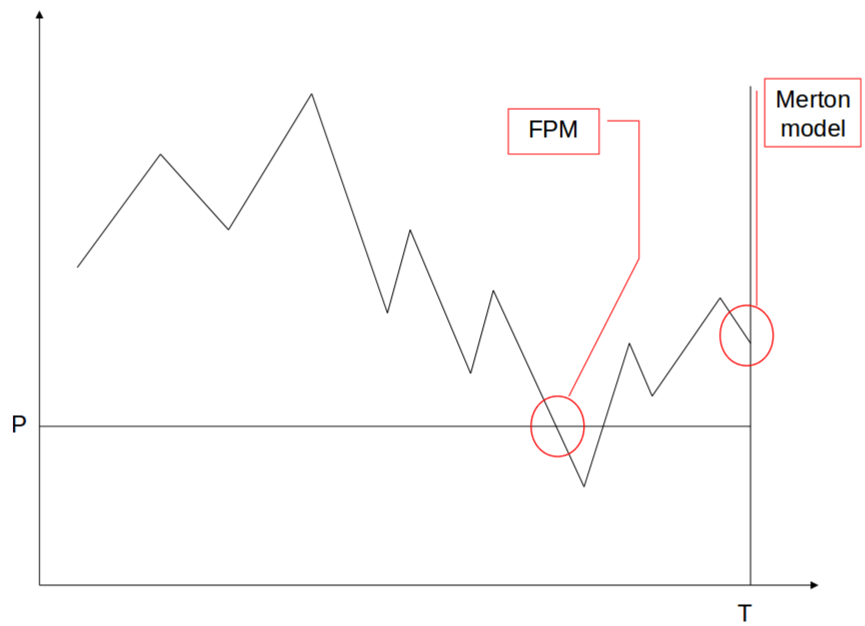

FIGURE 0.1. First Passage Model

hence

$$A_t = A_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z_t\right)$$

The first passage time τ is defined by

$$\begin{aligned}
\tau &= \inf\{t > 0 \,|\, A_t \le L\} \\
&= \inf\{t > 0 \,|\, A_t = L\} \\
&= \inf\left\{ t > 0 \,\middle|\, Z_t = \frac{\ln\left(\frac{A_t}{A_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)t}{\sigma} \right\} \\
&= \inf\{t > 0 \,|\, Z_t = a + bt\}
\end{aligned}$$

We are interested in the distribution of the first time that a Brownian motion hits a line.

1. Brownian Hitting Times

We can start with the distribution of the first time Brownian motion hits a level $a > 0$, i.e., $\tau_a = \inf\{t > 0 \,|\, Z_t = a\}$. Two important techniques: Stopping times and MGF's and Reflection Principle will be applied.

Recall that a random time τ is called a stopping time if we know whether or not $\tau \le t$ has occurred at time t. (It does not depend on the future). The optional sampling theorem extends the martingale property to stopping times. It says if M is a martingale and τ and σ are two **bounded** stopping times with $\sigma \le \tau$, then we have the following property

$$M_\sigma = \mathbb{E}\left[M_\tau \,|\mathcal{F}_\sigma\right]$$

Why boundedness is important? We can see that, in this example, $\sigma = 0$ and $\tau = \tau_a$

$$0 = Z_0 \ne \mathbb{E}\left[Z_\tau \,|\mathcal{F}_0\right] = \mathbb{E}\left[Z_\tau\right] = \mathbb{E}\left[a\right] = a$$

Recall the exponential martingales (it is obvious there is no drift term when applying Itô's lemma to M_t^λ), i.e.,

$$\begin{aligned}
M_t^\lambda &= \exp\left(-\frac{1}{2}\lambda^2 t + \lambda Z_t\right), \lambda > 0 \\
dM_t^\lambda &= \lambda \cdot M_t^\lambda dZ_t, M_0^\lambda = 1
\end{aligned}$$

Because τ is not bounded, we need to apply a trick to use Optional Sampling Theorem. Consider $\tau_n = \min\{\tau, n\}$, then we have

$$1 = M_0^\lambda = \mathbb{E}\left[M_{\tau_n}^\lambda\right]$$

and this choice of M allows us to interchange expectation and limit operators, i.e.,

$$\begin{aligned}
1 &= M_0^\lambda = \lim_{n\to\infty} \mathbb{E}\left[M_{\tau_n}^\lambda\right] \\
&= \mathbb{E}\left[\lim_{n\to\infty} M_{\tau_n}^\lambda\right] \\
&= \mathbb{E}\left[M_\tau^\lambda\right] \\
&= \mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}\tau + \lambda Z_\tau\right)\right] \\
&= \exp(\lambda a)\, \mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}\tau\right)\right]
\end{aligned}$$

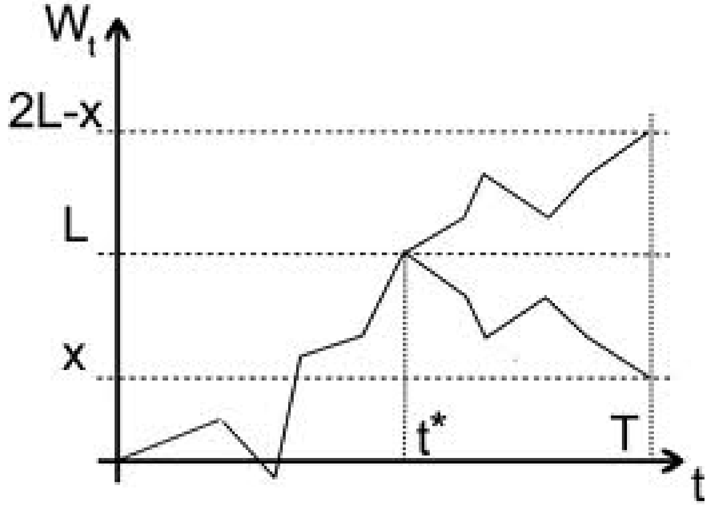

FIGURE 2.1. Reflection Principle

With obvious change of variable, we can get

$$\mathbb{E}\left[\exp\left(-c\tau\right)\right] = \exp\left(-a\sqrt{2c}\right)$$

Therefore the probability of hitting level a at some finite time

$$\begin{aligned}\mathbb{P}\left(\tau < \infty\right) &= \lim_{c\to 0}\mathbb{E}\left[\exp\left(-c\tau\right)\right] \\ &= \lim_{c\to 0}\exp\left(-a\sqrt{2c}\right) \\ &= 1\end{aligned}$$

and expectation on τ is given by (see this via MGF of τ)

$$\begin{aligned}\mathbb{E}\left[\tau\right] &= -\frac{d}{dc}\mathbb{E}\left[e^{-c\tau}\right]_{c=0} \\ &= -\frac{d}{dc}\exp\left(-a\sqrt{2c}\right)_{c=0} \\ &= \infty\end{aligned}$$

This phenomenon can be interpreted as: Z always hits a, but it might take a very long time to get there.

2. Reflection Principle

Personally I love the reflection principle so much. The very famous Ballot problem can be easily solved with this technique. If we start at $Z_0 = 0$ and τ is the first passage time to reach some positive level a, apparently we get

$$\mathbb{P}\left(\tau \le t\right) = \mathbb{P}\left(\max_{0\le s\le t} Z_s \ge a\right) = 2\mathbb{P}\left(Z_t \ge a\right) = 2\Phi\left(\frac{-a}{\sqrt{t}}\right)$$

If we differentiate, we get the hitting time distribution

$$f_a(t) = \frac{a}{\sqrt{2\pi \cdot t^3}} \exp\left(-\frac{a^2}{2t}\right)$$

For $a < 0$ case, just replace a in the above with $|a|$, i.e.,

$$f_a(t) = \frac{|a|}{\sqrt{2\pi \cdot t^3}} \exp\left(-\frac{a^2}{2t}\right)$$

3. Hitting Time Distribution to a Line

We can do the following transformation:

$$\begin{aligned} \tau &= \inf\{t > 0 \,|\, Z_t = a + bt\} \\ &= \inf\{t > 0 \,|\, Z_t - bt = a\} \\ &= \inf\{t > 0 \,|\, W_t = a\} \end{aligned}$$

where Z is a BM w.r.t. $\mathbb{P}$ and it suffices to find a $\mathbb{Q}$ such that W is a BM w.r.t. $\mathbb{Q}$

Girsanov Transformation says

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = M_t^b = \exp\left(-\tfrac{b^2}{2}t + bZ_t\right)$$
$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\left(\tfrac{b^2}{2}t - bZ_t\right) = \exp\left(-\tfrac{b^2}{2}t - bW_t\right)$$

where W is a $\mathbb{Q}$-Brownian motion.

We can conclude that

$$\begin{aligned} \mathbb{P}(\tau \le t) &= \mathbb{E}_{\mathbb{P}}\left[\mathbb{I}_{\{\tau \le t\}}\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\tau \le t\}} \cdot \exp\left(-\frac{b^2}{2}t - bW_t\right)\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\tau \le t\}} \cdot \exp\left(-\frac{b^2}{2}(t-\tau) - b(W_t - W_\tau)\right) \cdot \exp\left(-\frac{b^2}{2}\tau - bW_\tau\right)\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\tau \le t\}} \cdot \exp\left(-\frac{b^2}{2}\tau - bW_\tau\right)\right] \\ &= e^{-ba}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\tau \le t\}} \cdot \exp\left(-\frac{b^2}{2}\tau\right)\right] \end{aligned}$$

now we can get the density of the first hitting time to the line as the following

$$g_{a,b}(t) = \frac{|a|}{\sqrt{2\pi \cdot t^3}} \exp\left(-\frac{(a+bt)^2}{2t}\right)$$

Thus if we let t tend to infinity, the probability of hitting the line within a finite time is

$$\begin{aligned}
\mathbb{P}(\tau < \infty) &= \lim_{t\to\infty} \mathbb{P}(\tau \le t) \\
&= \lim_{t\to\infty} e^{-ba}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\{\tau\le t\}} \cdot \exp\left(-\frac{b^2}{2}\tau\right)\right] \\
&= e^{-ba}\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\frac{b^2}{2}\tau\right)\right] \\
&= e^{-ba} \cdot \exp\left(-|a|\sqrt{b^2}\right) \\
&= \exp(-ab - |ab|)
\end{aligned}$$

In conclusion, if a and b have the same sign, this probability is less than 1 otherwise it is equal to 1.

References

[1] Finance 3 Course Notes on "Credit Risk Part One: Structural Models", University of Waterloo 2008